

1/4-BPS M-theory bubbles with $SO(3) \times SO(4)$ symmetry

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ABSTRACT: In this paper we generalize the work of Lin, Lunin and Maldacena on the classification of 1/2-BPS M-theory solutions to a specific class of 1/4-BPS configurations. We are interested in the solutions of 11 dimensional supergravity with $SO(3) \times SO(4)$ symmetry, and it is shown that such solutions are constructed over a one-parameter family of 4 dimensional almost Calabi-Yau spaces. Through analytic continuations we can obtain M-theory solutions having $AdS_2 \times S^3$ or $AdS_3 \times S^2$ factors. It is shown that our result is equivalent to the AdS solutions which have been recently reported as the near-horizon geometry of M2 or M5-branes wrapped on 2 or 4-cycles in Calabi-Yau threefolds. We also discuss the hierarchy of M-theory bubbles with different number of supersymmetries.

KEYWORDS: D-branes, AdS-CFT Correspondence, Gauge-gravity correspondence, M-Theory.

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1. Introduction

We have recently witnessed that a systematic analysis of supersymmetric solutions in supergravity theories which utilises the existence of a Killing spinor can lead to a remarkable insight into string theory and strongly coupled gauge theory via the gauge/gravity correspondence [1]. Especially in [2], the authors consider 1/2-BPS fluctuations of maximally supersymmetric AdS solutions in type IIB supergravity and find that the entire set of regular solutions can be matched with the phase space of one-dimensional free fermions. This is in good harmony with the dual field theory, $\mathcal{N} = 4, D = 4$ super Yang-Mills model: in the 1/2-BPS sector it is reduced to a Hermitian matrix model whose eigenvalues can be treated as free fermions when one takes into account the Van der Monde determinant.

It is then natural to ask whether we can also identify the gauge dynamics for less supersymmetric operators from the geometric constraints imposed by unbroken supersymmetry on the supergravity side. While the 1/2-BPS solutions are equipped with $SO(4) \times SO(4)$ which results in $S^3 \times S^3$ part in the 10 dimensional metric, 1/4-BPS operators have $SO(4) \times SO(2)$ symmetry which implies that the solutions should have a $S^3 \times S^1$ factor. Supersymmetric solutions of type IIB supergravity with such isometries have been studied in [3, 4]. One can also consider 1/8-BPS solutions which have just a single S^3 factor in the metric [5], and it can be shown that the solution is constructed over a 6 dimensional

Kähler space obeying a type of non-linear Laplace equation for the Ricci tensor. See [6] for the study on a different class of 1/8-BPS solutions, and [7] for a unified viewpoint and a systematic analysis of supersymmetric regular solutions and the identification of the dual operators to 1/2, 1/4 and 1/8-BPS solutions.

It is also interesting to apply this program to M-theory. In [2] the authors considered the 1/2-BPS fluctuations, or bubbles, of M-theory as well and showed that the supergravity equations are reduced to a 3 dimensional continuous Toda equation. It is expected that this particular Toda equation is responsible for the dynamics of 1/2-BPS operators of the superconformal field theory defined on M2 or M5-branes, although in this case we do not have a perturbative description of the dual conformal field theory and it is not clear how to *derive* the Toda system from the field theory. See [8, 9] for discussions on the solutions of the Toda equation and their interpretations as giant gravitons.

One can try to determine the dual geometry for less-supersymmetric M-theory bubbles. 1/8-BPS solutions with an S^2 factor, or AdS_2 when analytically continued, has been studied already in [10] and the resulting BPS system satisfies, surprisingly enough, exactly the same equation - now defined in 8 dimensions - which governs S^3 bubbles of IIB theory. A natural interpretation of such configurations is that they are dual to BPS operators which are Lorentz singlet and holomorphic in $SU(4) \subset SO(8)$ R-symmetry of the M2-brane theory.

We are interested in 1/4-BPS bubbles of M-theory in this paper. If we consider the 6 dimensional field theory of M5-branes with $(2, 0)$ supersymmetry and restrict ourselves to BPS operators which are Lorentz-singlet but holomorphic in $SU(2) \subset SO(5)$ R-symmetry, the dual geometry should carry an $SO(6)$ symmetry which lead to an S^5 -factor in the metric. A related problem of supersymmetric AdS_5 solutions in M-theory has been addressed in [11] and the local geometry of the corresponding bubble solutions are obtained simply through analytic continuations. Although it will be very interesting to study the bubble solutions in detail and identify the dual operators, in this paper we restrict ourselves to the other class of 1/4-BPS M-theory bubbles. In M2-brane field theory, if a given Lorentz-singlet operator saturates the BPS bound and is written as a holomorphic combination of two chiral multiplets, it should be invariant under $SO(3) \times SO(4)$ symmetry so the dual geometry should contain $S^2 \times S^3$. We take this as our starting point and analyse how the supersymmetry helps us determine the local form of the solutions, filling the gap between the 1/2-BPS bubbles of [2] and the 1/8-BPS solutions of [10]. There exist a number of papers which explore the AdS/CFT relation using the supergravity backgrounds for specific M-brane configurations as duals to interesting field theory objects such as Wilson loops, defect conformal field theories etc. See for instance [12–14].

Once we establish the $S^2 \times S^3$ solutions, it is straightforward to obtain $AdS_2 \times S^3$ or $AdS_3 \times S^2$ via a series of analytic continuations. They are interpreted more naturally as the near-horizon geometry of (wrapped) M2 or M5-branes with some extra isometries in the transverse space. Such configurations have been already analysed using the supersymmetry condition of brane probes in the Calabi-Yau threefolds, by [15, 16]. We will show that our results indeed agree with the wrapped brane solutions.

Section 2 serves as the main part of this article. We first fix our convention and derive

the 6 dimensional Killing spinor equations in section 2.1. We then analyse the algebraic and differential equations of spinor bilinears to determine the geometry of the solutions in section 2.2. In section 3 we discuss how one can obtain the Wick-rotated versions $AdS_2 \times S^3$ and $AdS_3 \times S^2$ through analytic continuations, and show they are equivalent to the results of [15, 16]. In section 4 and section 5 we discuss how our solutions can be related to 1/2-BPS or 1/8-BPS M-theory bubbles from the literature. We conclude in section 6.

2. $S^2 \times S^3$ ansatz and the local form of the supersymmetric solutions

2.1 The metric ansatz and the Killing spinor equations in $D = 6$

In this paper, we aim to study supersymmetric solutions in 11 dimensional supergravity with $SO(3) \times SO(4)$ isometry which are dual to 1/4-BPS operators of the dual conformal field theory in 3 or 6 dimensions. We thus assume that the spacetime metric should contain $S^2 \times S^3$. More specifically, our ansatz is

$$ds_{11}^2 = e^{2A} ds_{S^2}^2 + e^{2B} ds_{S^3}^2 + g_{\mu\nu} dx^\mu dx^\nu, \tag{2.1}$$

$$G = F \wedge \text{Vol}_{S^2}, \tag{2.2}$$

where $ds_{S^2}^2$ and $ds_{S^3}^2$ represent the metric of the round sphere with radius 1 in the appropriate dimensionality. We dimensionally reduce the four-form field strength $G = dC$ to have a 6 dimensional gauge field F . Since electric(magnetic) configurations of G are associated to M2(M5)-branes, in our setting M2-branes are wrapped on S^2 and M5-branes contain the S^3 as part of the worldvolume.

We adopt the standard convention for the 11 dimensional supergravity with the lagrangian density

$$\mathcal{L} = R * 1 - \frac{1}{2} G \wedge * G - \frac{1}{6} C \wedge G \wedge G, \tag{2.3}$$

and the supersymmetry transformation for the gravitino is given as

$$\delta\psi_M = \nabla_M \epsilon + \frac{1}{288} \left(\Gamma_M^{M_1 \dots M_4} - 8 \delta_M^{M_1} \Gamma^{M_2 M_3 M_4} \right) G_{M_1 \dots M_4} \epsilon, \tag{2.4}$$

with the spinorial parameter ϵ which should obey the Majorana condition. Γ_M represents the 11 dimensional gamma matrices satisfying

$$\{\Gamma_M, \Gamma_N\} = 2g_{MN}, \tag{2.5}$$

where g_{MN} is the 11 dimensional metric tensor and $M, N = 0, 1, \dots, 10$.

Above ansatz can be understood as the dimensional reduction of 11 dimensional supergravity theory on (unsquashed) $S^2 \times S^3$, and we expect to have an effective action in 6 dimensions, which has the metric, two scalar fields A, B , and a two-form field strength F as the dynamical fields. It is worth noting here that in our ansatz the cubic Wess-Zumino term in (2.3) has no effect, so from the form-field equation and the Bianchi identity for 11 dimensional field we know F should satisfy simply

$$dF = 0, \tag{2.6}$$

$$d(e^{-2A+3B} *_6 F) = 0. \tag{2.7}$$

We need to choose a gamma matrix basis which respects the dimensional split we have introduced, to derive 6 dimensional Killing spinor equations from the 11 dimensional one. Our convention is, in Minkowski spacetime,

$$\begin{aligned}\Gamma_a &= \sigma_a \otimes 1 \otimes 1, & a &= 1, 2 \\ \Gamma_\alpha &= \sigma_3 \otimes \sigma_\alpha \otimes \gamma_7, & \alpha &= 1, 2, 3 \\ \Gamma_\mu &= \sigma_3 \otimes 1 \otimes \gamma_\mu, & \mu &= 0, 1, \dots, 5.\end{aligned}\tag{2.8}$$

where σ are the Pauli matrices. For simplicity we will choose the basis where the 6 dimensional gamma matrices γ_μ and γ_7 are all antisymmetric.

We can decompose an 11 dimensional Killing spinor as an expansion over the Killing spinors on S^2, S^3 , i.e.

$$\epsilon = \sum_i (\zeta_i \otimes \chi_i \otimes \eta_i + c.c.),\tag{2.9}$$

where $\zeta(\chi)$ is a 2(3) dimensional spinor, and η is the Killing spinor of the 6 dimensional system we are interested in. On the spheres S^2 and S^3 , the Killing spinor should be conformally parallel, which means

$$\bar{\nabla}_a \zeta = \pm \frac{1}{2} \sigma_a \sigma_3 \zeta,\tag{2.10}$$

$$\bar{\nabla}_\alpha \chi = \pm \frac{i}{2} \sigma_\alpha \chi,\tag{2.11}$$

where $\bar{\nabla}$ denotes the covariant derivative on the sphere with unit radius. For definiteness let us choose the positive sign in the above relations for ζ, χ . One can then derive the following 6 dimensional Killing spinor equations from $\delta\psi_M = 0$:

$$\left[\not{\partial} A - \frac{i}{6} e^{-2A} \not{F} + e^{-A} \right] \eta = 0,\tag{2.12}$$

$$\left[\not{\partial} B + \frac{i}{12} e^{-2A} \not{F} + i e^{-B} \gamma_7 \right] \eta = 0,\tag{2.13}$$

$$\nabla_\mu \eta - \frac{i}{48} e^{-2A} \gamma_\mu \not{F} \eta + \frac{i}{16} e^{-2A} \not{F} \gamma^\mu \eta = 0.\tag{2.14}$$

A comment is in order on different sign choices in (2.10) and (2.11) and the number of supersymmetries of our ansatz. The Killing spinors on the sphere should come in some irreducible representations of the isometry group. They make a doublet of $SU(2)$ for S^2 , and $(2, 1) \oplus (1, 2)$ of $SU(2) \times SU(2)$ for S^3 . For each of them, we expect to have a nontrivial solution to the 6 dimensional Killing spinor equation given above, so we should have 8 real solutions due to the Majorana condition in $D = 11$. Our ansatz thus should provide 1/4-BPS configurations in general.

2.2 Spinor bilinears and their properties

Let us now introduce the differential forms which are defined as spinor bilinears. We first

consider tensors whose components are given as $\bar{\eta}\gamma_{\mu_1\dots\mu_n}\eta$. Our convention goes as follows:

$$C = i\bar{\eta}\eta, \tag{2.15}$$

$$D = \bar{\eta}\gamma_7\eta, \tag{2.16}$$

$$K_\mu = \bar{\eta}\gamma_\mu\eta, \tag{2.17}$$

$$L_\mu = \bar{\eta}\gamma_\mu\gamma_7\eta, \tag{2.18}$$

$$Y_{\mu\nu} = \bar{\eta}\gamma_{\mu\nu}\eta, \tag{2.19}$$

$$Y'_{\mu\nu} = i\bar{\eta}\gamma_{\mu\nu}\gamma_7\eta, \tag{2.20}$$

$$Z_{\mu\nu\lambda} = i\bar{\eta}\gamma_{\mu\nu\lambda}\eta, \tag{2.21}$$

$$W_{\mu\nu\lambda\rho} = i\bar{\eta}\gamma_{\mu\nu\lambda\rho}\eta. \tag{2.22}$$

Note that they are all real-valued. One can of course also define additional tensors such as $Z'_{\mu\nu\lambda} = i\bar{\eta}\gamma_{\mu\nu\lambda}\gamma_7\eta$, but it is Poincare dual to Z . We will see shortly that the $D = 11$ solution is built upon a $D = 4$ Kahler space, so it is essentially the lower-rank tensors up to 2-forms which are needed to specify the local geometry of supersymmetric solutions.

Due to antisymmetry of γ_μ , tensors such as $\eta^T\gamma_7\eta, \eta^T\gamma_\mu\eta, \eta^T\gamma_\mu\gamma_7\eta, \eta^T\gamma_{\mu\nu}\eta$ vanish identically. We can easily see $\eta^T\eta = 0$ for nontrivial solutions from (2.12) or (2.13). We are thus left with the following tensors,

$$\omega_{\mu\nu} = \eta^T\gamma_{\mu\nu}\gamma_7\eta, \tag{2.23}$$

$$\phi_{\mu\nu\lambda} = \eta^T\gamma_{\mu\nu\lambda}\eta, \tag{2.24}$$

$$\psi_{\mu\nu\lambda\rho} = \eta^T\gamma_{\mu\nu\lambda\rho}\eta, \tag{2.25}$$

which are in general complex-valued.

Now we are ready to study the geometry of supersymmetric backgrounds using the existence of Killing spinors. We exploit the differential and algebraic constraints from the Killing equations and Fierz identities to identify the local form of the supersymmetric solutions.

Let us start with the scalars C, D . If we multiply $\bar{\eta}$ to (2.12) and (2.13),

$$e^{-A}C = 2e^{-B}D = -\frac{1}{6}e^{-2A}\bar{\eta}\not{F}\eta. \tag{2.26}$$

Furthermore, when we take the derivative of C , we get

$$\partial_\mu C = \frac{1}{12}e^{-2A}\bar{\eta}[\not{F}, \gamma_\mu]\eta \tag{2.27}$$

$$= \partial_\mu A C. \tag{2.28}$$

So, we fix the normalization of η and set

$$C = e^A, \quad D = e^B/2. \tag{2.29}$$

From now on we will make use of these relations whenever we come across C, D .

Now let us turn to the vectors. From the Fierz identity one can prove that

$$K \cdot L = 0, \quad (2.30)$$

$$K^2 + L^2 = 0. \quad (2.31)$$

and K is time-like, whereas L is space-like. One can also prove that in general

$$|\eta^T \eta|^2 = \frac{1}{2}(L^2 - K^2) - (C^2 + D^2). \quad (2.32)$$

But since $\eta^T \eta = 0$, we have

$$L^2 = -K^2 = e^{2A} + \frac{e^{2B}}{4}. \quad (2.33)$$

Readers are referred to appendix for details on Fierz rearrangement identities in 6 dimensions.

From the Killing spinor equations, it is straightforward to verify that

$$\nabla_{(\mu} K_{\nu)} = 0, \quad (2.34)$$

which implies K defines a Killing vector. And we can also see from the Killing spinor equation that the isometry of the metric associated with K is actually a symmetry of the whole solution. The Lie derivatives of scalar fields A, B and gauge field F all vanish. As a one-form, its exterior derivative is given as

$$d(e^A K) = F + Y. \quad (2.35)$$

For the other vector field L , from the algebraic relations we can derive

$$\begin{aligned} L_\mu &\equiv \bar{\eta} \gamma_\mu \gamma_\tau \eta \\ &= \frac{1}{2} e^{-B} \partial_\mu (e^{A+2B}), \end{aligned} \quad (2.36)$$

and from the differential Killing spinor equation (2.13),

$$\nabla_\mu L_\nu = -\frac{i}{48} e^{-2A} \bar{\eta} (\not{F} \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu \not{F}) \gamma_\tau \eta + \frac{i}{16} e^{-2A} \bar{\eta} (\gamma_\mu \not{F} \gamma_\nu + \gamma_\nu \not{F} \gamma_\mu) \gamma_\tau \eta, \quad (2.37)$$

leading to a significant requirement:

$$\nabla \cdot L = 0, \quad (2.38)$$

while the exterior derivative satisfies $d(e^{-A/2} L) = 0$, which is consistent with (2.36).

Now let us try to specify the 6 dimensional metric using the information we have collected so far. From the time-like Killing vector K , we introduce a time-like coordinate t and set $K = \partial_t$. With L we define a space-like coordinate as $y = e^{A/2+B}$ and set $L = e^{A/2} dy$. It will be convenient to define a scalar ζ as

$$\sinh \zeta = \frac{1}{2} y e^{-3A/2}, \quad (2.39)$$

to simplify the following discussions. In this coordinate system, we may write down the 6 dimensional metric as

$$ds_6^2 = -e^{2A} \cosh^2 \zeta \left(dt + \sum_{i=1}^4 V_i dx^i \right)^2 + \frac{e^{-A}}{\cosh^2 \zeta} dy^2 + e^{-A} \sum_{i,j=1}^4 h_{ij} dx^i dx^j. \quad (2.40)$$

Note that we have introduced a warp factor e^{-A} for the 4 dimensional space \mathcal{M}_4 with metric h_{ij} for later convenience.

The problem is now effectively reduced to 4 dimensions. When we introduce the gauge potential as $F = dB$, and expand

$$B = B_t dt + B_i dx^i + B_y dy, \quad (2.41)$$

we have the following unknown functions in 4 dimensions.

1. metric h_{ij}
2. scalars A, B_t, B_y
3. vectors V_i, B_i

and they all depend on the 5 dimensional spatial coordinates y, x^i in general.

Equipped with the local form of the metric, we are now in a position to choose an orthonormal frame. We set

$$e^0 = e^A \cosh \zeta (dt + V), \quad (2.42)$$

$$e^5 = e^{-A/2} \text{sech } \zeta dy, \quad (2.43)$$

$$e^i = e^{-A/2} \hat{e}^i, \quad i = 1, 2, 3, 4. \quad (2.44)$$

where \hat{e}^i is an orthonormal frame of the 4 dimensional metric h_{ij} .

Our system in general preserves 1/4 supersymmetry of the 11 dimensional supergravity, and the relevant projection rules can be best expressed using the orthonormal frame given above. From the algebraic Killing spinor equations we can eliminate the term with \not{F} to obtain

$$(\gamma_{\hat{5}} \cosh \zeta + \sinh \zeta + i\gamma_{\hat{7}}) \eta = 0, \quad (2.45)$$

where the gamma matrices with hatted indices are defined in the tangent space. We can simplify (2.45) in terms of $\tilde{\eta} = e^{\zeta/2\gamma_{\hat{5}}} \eta$ and obtain

$$(1 + i\gamma_{\hat{5}}\gamma_{\hat{7}})\tilde{\eta} = 0. \quad (2.46)$$

Considering $L = e^A \cosh \zeta e^5$, one can find the other projection condition

$$(1 - i\gamma_{\hat{0}})\tilde{\eta} = 0, \quad (2.47)$$

and the normalization of $\tilde{\eta}$,

$$\tilde{\eta}^\dagger \tilde{\eta} = e^A. \quad (2.48)$$

The projection rules imply that $\tilde{\eta}$ is a chiral spinor in \mathcal{M}_4 . As it is well known, an invariant Weyl spinor in $2n$ -dimensional space defines an $SU(n)$ -structure, and the intrinsic torsion can be inferred from the derivatives of the invariant tensors which are constructed as spinor bilinears [17, 18].

The 4 dimensional $SU(2)$ -invariant tensors are included in the 6 dimensional spinor bilinears we have constructed, and we only need to see how the 6 dimensional tensors are decomposed into 4 dimensions. One can either directly evaluate each component of the tensors using (2.46), (2.47) and (2.48), or make use of the appropriate Fierz identities. Recall first it is our convention that

$$K = -e^{2A} \cosh^2 \zeta (dt + V), \tag{2.49}$$

$$L = e^{A/2} dy. \tag{2.50}$$

From (B.11) and (B.12),

$$Y = \frac{1}{2}(dt + V) \wedge y dy + J, \tag{2.51}$$

where $J = \frac{1}{2}J_{ij}dx^i \wedge dx^j$ is a 2-form in 4 dimensions which may have a nontrivial dependence on y . The higher-rank tensors turn out to be products of one- and two-forms given above. One can also easily see that the 3-form Z can be written as

$$Z = -e^{-A}K \wedge Y \tag{2.52}$$

$$= e^A \cosh^2 \zeta (dt + V) \wedge J. \tag{2.53}$$

and the 4-form W is

$$W = -\frac{1}{2}e^{-A}Y \wedge Y \tag{2.54}$$

$$= -\frac{1}{2}e^{-A}J \wedge J - (dt + V) \wedge y dy \wedge J. \tag{2.55}$$

One can also consider the complex-valued 2-form ω and find it is a 2-form purely in \mathcal{M}_4 as one can readily see from (B.16) and (B.17). In addition to that, we have

$$\phi = -e^{-A/2} \left(\frac{y}{2}(dt + V) + i \frac{dy}{\cosh^2 \zeta} \right) \wedge \omega, \tag{2.56}$$

$$\psi = e^{A/2}(dt + V) \wedge dy \wedge \omega. \tag{2.57}$$

From the direct evaluation or the normalization properties such as (B.9) and (B.10), we see that J can be used to define an almost complex structure with metric h_{ij} , and $\Omega = (\text{sech } \zeta) \cdot \omega$ provides the properly normalized $(2,0)$ -form, satisfying

$$\Omega \wedge J = 0, \quad \text{Vol}(\mathcal{M}_4) = \frac{1}{4}\Omega \wedge \bar{\Omega} = \frac{1}{2}J \wedge J. \tag{2.58}$$

The 6 dimensional derivatives can be decomposed with respect to our coordinate choice, so we can write

$$d = d_4 + dy \wedge \partial_y + dt \wedge \partial_t. \tag{2.59}$$

We now resume the computation of exterior derivatives for our spinor bilinears. Again employing the algebraic and differential Killing spinor equation, one easily obtains

$$dY = 0. \tag{2.60}$$

When rephrased in 4 dimensional language, it implies

$$d_4 J = 0, \tag{2.61}$$

$$\partial_y J = -\frac{y}{2} d_4 V, \tag{2.62}$$

$$\partial_t J = 0. \tag{2.63}$$

One can also see that

$$d\omega = 0, \tag{2.64}$$

which implies

$$d_4(\cosh \zeta \cdot \Omega) = 0, \tag{2.65}$$

$$\partial_y(\cosh \zeta \cdot \Omega) = 0, \tag{2.66}$$

$$\partial_t(\cosh \zeta \cdot \Omega) = 0. \tag{2.67}$$

The 4 dimensional derivatives of the SU(2) tensors (2.61) and (2.65) determine the SU(2) structure and the intrinsic torsion of \mathcal{M}_4 . From the fact that the (1,1) form J is d_4 -closed, and there exists a (2,0) form $\cosh \zeta \Omega$ which is also closed, we conclude that \mathcal{M}_4 is almost Calabi-Yau [19]. One notes that (2.65) can be expressed as

$$d_4 \Omega = iP \wedge \Omega, \tag{2.68}$$

with

$$P = \frac{3}{2} \tanh^2 \zeta (J \cdot dA). \tag{2.69}$$

As it is well-known, P is the Ricci potential whose exterior derivative gives the Ricci form $\mathfrak{R} = dP$.

For higher-rank tensors, after similar manipulations we obtain

$$d(e^{2A} Z) = 2e^A W - F \wedge Y, \tag{2.70}$$

$$d(e^A W) = 0, \tag{2.71}$$

$$d(e^{A/2} \phi) = \frac{1}{2} e^{-A/2} \psi. \tag{2.72}$$

We can check that these equations automatically hold once we demand the supersymmetry conditions given in previous paragraphs. It is basically because these higher-rank tensors are expressed as exterior products of 1 and 2-forms, as given in (2.52), (2.54), (2.56), and (2.57), and do not pose genuinely new invariant tensors.

The gauge field F can be determined by (2.35) once the geometric data and scalar field A are given. When decomposed into 4 dimensions we have

$$F = -d(e^{3A}) \wedge (dt + V) + e^{3A} \cosh^2 \zeta \partial_y V \wedge dy + \hat{F}, \tag{2.73}$$

where \hat{F} represents the 4 dimensional part of F and is given as

$$\hat{F} = -J - e^{3A} \cosh^2 \zeta d_4 V. \tag{2.74}$$

At this stage, we can make use of the algebraic Killing equation (2.13) to derive various constraints on F . In particular, we consider (A.2), (A.5) and find the following relations,

$$(d_4 V)_+ = \frac{2}{y} \partial_y (\ln \cosh \zeta) J, \tag{2.75}$$

$$\partial_y V = -\frac{6 \sinh^2 \zeta}{y \cosh^4 \zeta} J \cdot dA. \tag{2.76}$$

We now see that the Ricci potential P can be written more succinctly as

$$P = -\frac{y}{4} \cosh^2 \zeta \partial_y V, \tag{2.77}$$

and when we take d_4 ,

$$y \partial_y \left(\frac{1}{y} \partial_y J \right) = d_4 (J \cdot d \operatorname{sech}^2 \zeta). \tag{2.78}$$

This equation can be considered as a higher dimensional analogue of the Toda equation for the 1/2-BPS fluctuations considered in [2]. 1/4-BPS bubbles of IIB supergravity satisfies a very similar differential equation, see (58) of [3].

We are now in a position to check whether our supersymmetric configurations described so far automatically satisfy the classical field equations. As well-established by now, supersymmetry requirements combined with the Bianchi identity and the form-field equations imply that the Einstein equation is satisfied, unless the Killing spinor is null [20]. For the solutions of our interest in this paper, $K^2 = -L^2 < 0$ so the Killing spinor is not null. From the equations (2.35) and (2.51) it follows that

$$dF = 0. \tag{2.79}$$

Now let us check the field equation (2.7). Among the various supersymmetry requirement conditions, we use (2.35), (A.7), (2.52) and (2.54) to obtain an expression for $*F$ in terms of the geometric data including A .

$$*F = e^{-B} Y \wedge Y + 2e^{3A-B} d(e^{-2A} K \wedge Y). \tag{2.80}$$

Now we can check (2.7) using the 4 dimensional decompositions of K, Y given in (2.49), (2.60). It is straightforward to see that it vanishes provided $\partial_y (\cosh^2 \zeta J \wedge J) = 0$. But this is a consequence of (2.38), or equivalently (2.65). So we have now established that the equations of motion are satisfied for our supersymmetric configurations.

2.3 Summary of the result

Let us give a summary of our results in this short subsection for easier reference. One can rephrase our results in a more concise way as follows. The 1/4-BPS solutions of M-theory

with $S^2 \times S^3$ can be always written, up to a coordinate transformation, as follows.

$$ds_{11}^2 = \left(\frac{y}{2 \sinh \zeta} \right)^{4/3} \left[ds_{S^2}^2 + 4 \sinh^2 \zeta ds_{S^3}^2 + \frac{4 \sinh^2 \zeta}{y^2} \left(- \frac{y^2}{4 \tanh^2 \zeta} (dt + V)^2 + \frac{dy^2}{\cosh^2 \zeta} + h_{ij} dx^i dx^j \right) \right], \quad (2.81)$$

$$G = \text{Vol}_{S^2} \wedge \left(\frac{y}{2} dy \wedge (dt + V) - \frac{1}{4} d \left[y^2 \coth^2 \zeta (dt + V) \right] - J \right). \quad (2.82)$$

The 4-dimensional base space with metric h_{ij} is almost Calabi-Yau, i.e. its $SU(2)$ structure is dictated as

$$d_4 J = d_4 (\cosh \zeta \Omega) = 0, \quad (2.83)$$

which implies that the Ricci potential is given as

$$P = -J \cdot d_4 \ln \cosh \zeta. \quad (2.84)$$

The y -dependence of the 4-dimensional space is described as

$$\partial_y J = -\frac{y}{2} d_4 V, \quad \partial_y (\cosh \zeta \Omega) = 0. \quad (2.85)$$

It turns out that the supersymmetry also requires that the Ricci potential has an alternative expression

$$P = -\frac{y}{4} \cosh^2 \zeta \partial_y V, \quad (2.86)$$

and the integrability condition for V then leads to the following equation.

$$y \partial_y \left(\frac{1}{y} \partial_y J \right) = d_4 (J \cdot d \text{sech}^2 \zeta). \quad (2.87)$$

3. Analytic continuation to $AdS_2 \times S^3$ and $AdS_3 \times S^2$

We have so far considered a specific class of supersymmetric solutions in D=11 supergravity: configurations with an $S^2 \times S^3$ factor. If one is interested in similar problems, for instance M-theory solutions with $AdS_2 \times S^3$, obviously the same technique can be used to first derive the 6 dimensional Killing spinor equations and then study the local form of the metric and form-fields constrained by unbroken supersymmetries. But since we are interested in solutions containing a product of maximally symmetric spaces with the same dimensionalities, we can simply take multiple analytic continuations to transform our results on $S^2 \times S^3$ to $AdS_2 \times S^3$ or $AdS_3 \times S^2$. Actually, such new solutions might have even more significance in general. $AdS_2 \times S^3$ is the near-horizon geometry of 5 dimensional black holes, so the general form of the metric can be very useful in the systematic study of 5 dimensional supersymmetric black holes embedded in 11 dimensional supergravity. $AdS_3 \times S^2$ solutions are potentially dual to 2 dimensional supersymmetric conformal field theory whose R-symmetry has an $SU(2)$ factor.

Alternatively, one can also interpret the AdS solutions as a near-horizon limit of M2 or M5-branes. If one recalls that in our ansatz we have dimensionally reduced the 4-form field of the 11 dimensional supergravity on S^2 , one can easily conclude that the

$AdS_2 \times S^3$ solutions are purely M2-brane configurations while the $AdS_3 \times S^2$ solutions are composed purely of M5-branes. Since we have 1/4-BPS solutions, we can consider either intersection of two M-branes, or M-branes wrapped on supersymmetric cycles of Calabi-Yau 3 manifolds, to obtain the desired solutions.

In fact, AdS solutions as near-horizon limits of wrapped M-branes have been systematically studied recently, first for M5-branes in [15] and also for M2-branes in [16]. The authors used the fact that the Killing spinors of the supergravity configurations should obey the same projection rule required for the probe brane action, and found the local form of the solutions in an efficient way using the calibration conditions. $AdS_3 \times S^2$ solutions are given in (6.8-6.15) of [15], and $AdS_2 \times S^3$ solutions given in (4.12-4.19) of [16]. One can check that these AdS solutions are exactly the same as our solution, albeit written in different variables. Here we briefly sketch how to establish the equivalence of [15] and our results. A similar relation can be also easily found with $AdS_2 \times S^3$ solutions of [16]. It is useful first to note that the triplet of almost complex structures J^1, J^2, J^3 which describe the 4-dimensional base space in [15] are translated in our convention as $J^1 \rightarrow e^{-A}J, J^2 + iJ^3 \rightarrow e^{-A}\Omega$. Now it is straightforward to check that (6.10) and (6.11) in [15] correspond to (2.64). Similarly, (6.12) of [15] is equivalent to (2.60). In particular, when we complexify (6.13) and (6.14), the resulting equation is equivalent to (2.72).

In the rest of this subsection we illustrate how one can analytically continue $S^2 \times S^3$ solutions to obtain AdS solutions, and write the form of the metric for easier reference. By analytic continuation we mean we set all the coordinates of the round sphere to pure imaginary. For instance, start with the 2-sphere with metric

$$ds^2(S^2) = d\theta^2 + \sin^2\theta d\phi^2, \tag{3.1}$$

and through the reparametrization $\theta = i\rho, \phi = i\tau$, the metric becomes

$$ds^2 = -d\rho^2 + \sinh^2\rho d\tau^2 \tag{3.2}$$

$$= -ds^2(AdS_2). \tag{3.3}$$

To fix the overall sign of the metric, we further take the re-definition $e^{2A} \rightarrow -e^{2A}$ but leave e^{2B} invariant, or $y^2 \rightarrow iy^2$. In particular, now the metric can be written as

$$ds^2 = e^{2A} ds_{AdS_2}^2 + y^2 e^{-A} ds_{S^3}^2 + e^{2A} \cos^2\zeta (d\psi + V)^2 + \frac{e^{-A}}{\cos^2\zeta} dy^2 + e^{-A} h_{ij} dx^i dx^j, \tag{3.4}$$

where we introduced a space-like coordinate ψ by setting $t \rightarrow \psi$. ζ is defined as

$$\sin\zeta = \frac{1}{2} y e^{-3A/2}, \tag{3.5}$$

so for consistency the range of y is restricted to satisfy $\sin^2\zeta \leq 1$, unlike the $S^2 \times S^3$ solutions.

It is also straightforward to consider $AdS_3 \times S^2$. The metric can be written as

$$ds^2 = e^{2A} ds_{S^2}^2 + y^2 e^{-A} ds_{AdS_3}^2 + \frac{y^2 e^{-A}}{4} \cos^2\xi (d\psi + V)^2 + \frac{4e^{2A}}{y^2 \cos^2\xi} dy^2 + e^{-A} h_{ij} dx^i dx^j, \tag{3.6}$$

with

$$\sin\xi = \frac{2}{y} e^{3A/2}. \tag{3.7}$$

4. Examples and the identification of Kähler spaces

The most prominent examples of M-theory solution with $S^2 \times S^3$ are certainly the maximally supersymmetric configurations $AdS_4 \times S^7$ and $AdS_7 \times S^4$. For definiteness here we consider $AdS_4 \times S^7$ and re-write the metric in a way compatible with our results in this paper. The other case of $AdS_7 \times S^4$ can be treated in a similar way.

Let us first start with the 11 dimensional metric, which can be written as follows to make $SO(3) \times SO(4)$ isometry manifest.

$$ds_{11}^2 = R^2 [d\rho^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho ds_2^2 + 4(d\theta^2 + \sin^2 \theta ds_3^2 + \cos^2 \theta d\tilde{s}_3^2)]. \quad (4.1)$$

Let us choose ds_2^2 and ds_3^2 as the part corresponding to our S^2 and S^3 . Obviously, we can identify as

$$e^{2A} = \sinh^2 \rho, \quad (4.2)$$

$$e^{2B} = 4 \sin^2 \theta, \quad (4.3)$$

so

$$\sinh \zeta = \frac{\sin \theta}{\sinh \rho}. \quad (4.4)$$

In order to identify the 4 dimensional locally Kähler space, we split the metric of S^3 using the left-invariant forms of $SU(2)$.

$$d\tilde{s}_3^2 = \frac{1}{4} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \quad (4.5)$$

$$= \frac{1}{4} [(d\psi + \alpha)^2 + \sigma_1^2 + \sigma_2^2], \quad (4.6)$$

where $d\alpha = \sigma_1 \wedge \sigma_2$. Now if we take the re-parametrization $\psi \rightarrow \psi + t$ the 6 dimensional part of the metric becomes

$$ds_6^2 = -(\sinh^2 \rho + \sin^2 \theta) \left[dt - \frac{\cos^2 \theta}{\sinh^2 \rho + \sin^2 \theta} \sigma_3 \right]^2 + \frac{\cosh^2 \rho \cos^2 \theta}{\sinh^2 \rho + \sin^2 \theta} \sigma_3^2 + d\rho^2 + 4d\theta^2 + \cos^2 \theta (\sigma_1^2 + \sigma_2^2). \quad (4.7)$$

First of all we can now see the identification

$$V = -\frac{\cos^2 \theta}{\sinh^2 \rho + \sin^2 \theta} \sigma_3. \quad (4.8)$$

In order to identify the 4 dimensional Kähler part which is transverse to K, L vectors, it is required to compute $\frac{e^{-A}}{\cosh^2 \zeta} dy^2$ part of (2.40) and subtract it from (4.7). Upon the change of coordinates

$$y = 2\sqrt{\sinh \rho} \sin \theta, \quad (4.9)$$

$$v = 2\sqrt{\cosh \rho} \cos \theta, \quad (4.10)$$

it is straightforward to check

$$d\rho^2 + 4d\theta^2 = \frac{\sinh \rho}{\sinh^2 \rho + \sin^2 \theta} dy^2 + \frac{\cosh \rho}{\sinh^2 \rho + \sin^2 \theta} dv^2, \quad (4.11)$$

where ρ, θ are now treated as functions of y, v implicitly through the inversion of (4.9), (4.10). Now we can write down the metric of \mathcal{M}_4 which is expected to have a locally Kähler structure.

$$ds_{\mathcal{M}_4}^2 = \sinh \rho \left[\frac{\cosh \rho}{\sinh^2 \rho + \sin^2 \theta} dv^2 + \frac{\cosh^2 \rho \cos^2 \theta}{\sinh^2 \rho + \sin^2 \theta} \sigma_3^2 + \cos^2 \theta (\sigma_1^2 + \sigma_2^2) \right]. \quad (4.12)$$

The conditions on the SU(2)-structure of \mathcal{M}_4 , such as the equations which are derived from $dY = d\omega = 0$, can be verified once we fix the complex structure, or the Kähler form of \mathcal{M}_4 . It turns out that we need to choose

$$J = \frac{\sinh \rho \cosh^{3/2} \rho \cos \theta}{\sinh^2 \rho + \sin^2 \theta} dv \wedge \sigma_3 + \sinh \rho \cos^2 \theta \sigma_1 \wedge \sigma_2, \quad (4.13)$$

then it is straightforward to check that

$$d_4 J = 0, \quad (4.14)$$

$$\partial_y J = -\frac{y}{2} d_4 V, \quad (4.15)$$

indeed hold. The rest of the constraints can be also shown to be satisfied.

We next consider the 1/2-BPS bubble solutions of M-theory obtained in [2]. The relevant little group of the supersymmetric states is $SO(3) \times SO(6)$, which should appear as $S^2 \times S^5$ within the dual geometry. The Killing spinor analysis has been performed in [2] and we quote the result here,

$$ds^2 = -4e^{2\lambda}(1 + \tilde{y}^2 e^{-6\lambda})(dt + \tilde{V}_i dx^i)^2 + \frac{e^{-4\lambda}}{1 + \tilde{y}^2 e^{-6\lambda}} [d\tilde{y}^2 + e^D(dx_1^2 + dx_2^2)] + 4e^{2\lambda} ds^2(S^5) + \tilde{y}^2 e^{-4\lambda} ds^2(S^2), \quad (4.16)$$

$$G = \text{Vol}(S^2) \wedge F, \quad (4.17)$$

$$e^{-6\lambda} = \frac{\partial_y D}{y(1 - y\partial_y D)}, \quad (4.18)$$

$$\tilde{V}_i = \frac{1}{2} \epsilon_{ij} \partial_j D, \quad (4.19)$$

$$F = dB_t \wedge (dt + \tilde{V}) + B_t d\tilde{V} + d\hat{B}, \quad (4.20)$$

$$B_t = -4\tilde{y}^3 e^{-6\lambda}, \quad (4.21)$$

$$d\hat{B} = 2 *_3 [(y\partial_y^2 D + y(\partial_y D)^2 - \partial_y D) dy + y\partial_i \partial_y D dx^i]. \quad (4.22)$$

The scalar function D satisfies a 3 dimensional version of the Toda equation

$$(\partial_1^2 + \partial_2^2)D + \partial_y^2 e^D = 0. \quad (4.23)$$

In order to identify the 4 dimensional almost Calabi-Yau space as a verification of our result, we first write S^5 as a fibration over S^3 ,

$$d\Omega_5^2 = d\alpha^2 + \cos^2 \alpha d\psi^2 + \sin^2 \alpha ds^2(S^3), \quad (4.24)$$

Obviously one can identify

$$e^{2A} = \tilde{y}^2 e^{-4\lambda}, \quad (4.25)$$

$$e^{2B} = 4e^{2\lambda} \sin^2 \alpha. \quad (4.26)$$

In order to identify the 4 dimensional Kähler base, we first shift $\psi \rightarrow \psi + t$, and introduce a new set of coordinates (y, v, z_1, z_2) from $(\tilde{y}, \alpha, x_1, x_2)$ as follows

$$y = 2\sqrt{\tilde{y}} \sin \alpha, \quad (4.27)$$

$$u = e^{D/2} \cos \alpha, \quad (4.28)$$

$$z_1 = x_1, \quad (4.29)$$

$$z_2 = x_2. \quad (4.30)$$

Then one can show that the metric tensor becomes

$$\begin{aligned} ds_{11}^2 &= \tilde{y}^2 e^{-4\lambda} ds^2(S^2) + 4e^{2\lambda} \sin^2 \alpha ds^2(S^3) \\ &\quad - 4(e^{2\lambda} \sin^2 \alpha + \tilde{y}^2 e^{-4\lambda}) \left[dt + \frac{(1 + \tilde{y}^2 e^{-6\lambda})\tilde{V} - \cos^2 \alpha d\psi}{\sin^2 \alpha + \tilde{y}^2 e^{-6\lambda}} \right]^2 + \frac{\tilde{y} e^{-4\lambda}}{\sin^2 \alpha + \tilde{y}^2 e^{-6\lambda}} dy^2 \\ &\quad + \tilde{y}^{-1} e^{2\lambda} \left\{ 4\tilde{y} \cos^2 \alpha \frac{1 + \tilde{y}^2 e^{-6\lambda}}{\sin^2 \alpha + \tilde{y}^2 e^{-6\lambda}} (d\psi - \tilde{V})^2 + \frac{\tilde{y} e^{-6\lambda}}{1 + \tilde{y}^2 e^{-6\lambda}} e^D (dz_1^2 + dz_2^2) \right. \\ &\quad \left. + 4\tilde{y} e^{-D} \frac{1 + \tilde{y}^2 e^{-6\lambda}}{\sin^2 \alpha + \tilde{y}^2 e^{-6\lambda}} \left[du + e^{D/2} \cos \alpha (\tilde{V}_2 dz_1 - \tilde{V}_1 dz_2) \right]^2 \right\}. \end{aligned} \quad (4.31)$$

We choose the Kähler form as

$$\begin{aligned} J &= 4\tilde{y} \cos \alpha e^{-D/2} \frac{1 + \tilde{y}^2 e^{-6\lambda}}{\sin^2 \alpha + \tilde{y}^2 e^{-6\lambda}} \left(du + e^{D/2} \cos \alpha (\tilde{V}_2 dz_1 - \tilde{V}_1 dz_2) \right) \wedge (d\psi - \tilde{V}) \\ &\quad - \frac{\tilde{y} e^{-6\lambda}}{1 + \tilde{y}^2 e^{-6\lambda}} e^D dz_1 \wedge dz_2, \end{aligned} \quad (4.32)$$

and one can check that $d_4 J = 0$ and $\partial_y J = -\frac{y}{2} d_4 V$, using (4.23). The $(2, 0)$ -form Ω is taken as follows,

$$\begin{aligned} \Omega &= e^{i\psi} \left(4\tilde{y} \frac{\tilde{y} e^{-6\lambda}}{\sin^2 \alpha + \tilde{y}^2 e^{-6\lambda}} \right)^{1/2} \\ &\quad \left[(du + e^{D/2} \cos \alpha (\tilde{V}_2 dz_1 - \tilde{V}_1 dz_2)) + i(d\psi - \tilde{V}) \right] \wedge (dz_1 - idz_2), \end{aligned} \quad (4.33)$$

which satisfies

$$d_4(\cosh \zeta \Omega) = 0, \quad \partial_y(\cosh \zeta \Omega) = 0. \quad (4.34)$$

5. The relation to 1/8-BPS AdS bubbles

Supersymmetric M2-brane configurations with an AdS_2 factor in the metric has been studied in [10]. It turns out that the 9 dimensional internal space should take the form of a

warped U(1)-fibration over an 8 dimensional Kähler space \mathcal{M}_8 . One can also easily translate the results into the case of solutions with an S^2 , instead of AdS_2 , through analytic continuation. They would in general provide 1/8-BPS bubbles of M-theory. The specific type of solutions with $S^2 \times S^3$ we have studied in this paper can be considered as a special case of such 1/8-BPS solutions. In this section we show how the 1/4-BPS solutions studied in this paper can be re-written in a way as presented in [10, 21].

Let us first briefly summarize the result of [10]. One starts with the following ansatz:

$$ds^2 = e^{2\bar{A}} [ds^2(S_2) + ds^2(Y_9)], \quad (5.1)$$

$$G = \text{Vol}(S^2) \wedge F. \quad (5.2)$$

The existence of a nontrivial Killing spinor restricts the local form of the solution as follows,

$$ds^2(Y_9) = -(dt + P)^2 + e^{-3\bar{A}} ds^2(\mathcal{M}_8), \quad (5.3)$$

$$F = J + d[e^{4\bar{A}}(dt + P)]. \quad (5.4)$$

\mathcal{M}_8 is required to be Kähler with Kähler form J , and Ricci potential P . The warp factor is also determined purely by the geometric data of \mathcal{M}_8 ,

$$e^{-3\bar{A}} = -\frac{1}{2}R. \quad (5.5)$$

The Einstein equation combined with the supersymmetry requirement demands that the Ricci tensor of \mathcal{M}_8 should satisfy the following equation.

$$\square R - \frac{1}{2}R^2 + R_{mn}R^{mn} = 0. \quad (5.6)$$

One can construct new AdS_2 (or S^2) solutions in 11 dimensional supergravity based on a solution of (5.6). Indeed, a countably infinite number of new AdS_2 solutions in M-theory have been obtained in [21] using a co-homogeneity 1 solution of (5.6).

Let us now try to identify the 8 dimensional space from the result we obtained in this paper. Obviously we first identify the two S^2 's in (2.40) and (5.1) and set $\bar{A} = A$. We then write the metric of S^3 explicitly using the left-invariant forms of SU(2)

$$ds^2(S^3) = \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2). \quad (5.7)$$

Now upon a coordinate shift $\sigma_3 \rightarrow \sigma_3 + dt$, we can re-arrange the metric (2.40) into a form found in (5.1) and (5.3), and identify the metric of 8 dimensional Kähler base as

$$ds^2(\mathcal{M}_8) = \text{sech}^2 \zeta dy^2 + \frac{y^2}{4} \cosh^2 \zeta (\sigma_3 - V)^2 + \frac{y^2}{4} (\sigma_1^2 + \sigma_2^2) + \sum_{i,j=1}^4 h_{ij} dx^i dx^j. \quad (5.8)$$

And the Ricci potential is given as

$$P = V \cosh^2 \zeta - \sinh^2 \zeta \sigma_3. \quad (5.9)$$

In order to check the consistency conditions we introduce the Kähler form of \mathcal{M}_8 as

$$J_8 = \frac{y}{2} dy \wedge (\sigma_3 - V) + \frac{y^2}{4} \sigma_1 \wedge \sigma_2 + J_4, \quad (5.10)$$

where J_4 denotes the Kähler form of \mathcal{M}_4 . One can easily check $dJ_8 = 0$ using $\partial_y J_4 + \frac{y}{2} d_4 V = 0$. The associated $(4, 0)$ -form is given as

$$\Omega_8 = \left(\frac{y}{2} \operatorname{sech} \zeta dy + i \frac{y^2}{4} \cosh \zeta (\sigma_3 - V) \right) \wedge (\sigma_1 + i \sigma_2) \wedge \Omega_4. \quad (5.11)$$

It is also straightforward to check $d\Omega_8 = iP \wedge \Omega_8$ with P given as (5.9), so dP indeed gives the Ricci-form of \mathcal{M}_8 .

6. Discussions

In this paper we have used the technique of Killing spinor analysis to determine the geometric constraints imposed by the requirement of supersymmetry and $SO(3) \times SO(4)$ isometry in M-theory. The main motivation for this work has been to generalize the AdS bubble solutions of [2] to 1/4-BPS solutions. Like other examples of supersymmetric AdS bubbles reported earlier in [3–5, 10], it turns out that the 11 dimensional spacetime is based on a Kähler subspace. It is natural to associate this symplectic structure with the phase space of the gauge field dynamics for the BPS sector. We have derived a set of partial differential equations which determines the Kähler base space and eventually the 11 dimensional metric and the gauge field. Technically the partial differential equations can be derived if one first considers $AdS_2 \times S^3$ or $AdS_3 \times S^2$ and continue analytically to $S^2 \times S^3$ case. The relevant AdS solutions have been already studied in [15] and [16]. We argued that all of them essentially lead to the same equations, in section 3.

Once we have reduced all the equations of motion in 11 dimensions down to 5 dimensions spanned by y, x^i , the next step is to solve the equations like (2.78) and obtain new solutions. We leave this task for future publications, and put more emphasis on the hierarchy of Kähler spaces associated with different types of AdS bubbles. 1/2-BPS bubbles of [2], including the maximally supersymmetric solutions, provide nontrivial solutions of (2.78). In turn, the solutions presented in this paper would automatically satisfy another highly nontrivial equation (5.6) which describes the dynamics of 1/8-BPS bubbles.

It is also very important to find the connection of our results with the dual field theory dynamics. For 1/2-BPS bubbles of IIB theory, where the field theory is amenable to perturbative analysis since it is reduced to a hermitian matrix model, there has been considerable progress in relating the Yang-Mills theory with the semiclassical treatment of IIB supergravity theory [22–27]. See also [28] for analogous discussions on bubbles of $AdS_3 \times S^3$. Together with the insight one earns from the concrete computations on both sides of the duality in the above works, we hope that our results on the geometry of supergravity solutions play an important role in uncovering the microscopic building block of the dual conformal field theory on M-branes.

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A. Algebraic relations between the spinor bilinears

A number of algebraic equations can be derived for the spinor bilinears from the algebraic Killing equations (2.12) and (2.13). We first start with (2.12) and take contractions with $\bar{\eta}$ after multiplying different numbers of gamma matrices. If we multiply $\bar{\eta}$ we have

$$K^\mu \partial_\mu A = 0, \quad (\text{A.1})$$

$$F_{\mu\nu} Y^{\mu\nu} + 6e^{2A} = 0. \quad (\text{A.2})$$

And if we take contractions with $\bar{\eta}\gamma_\mu$ to get one-form equations we obtain

$$\partial_\mu A + \frac{1}{3}e^{-2A}F_{\mu\nu}K^\nu = 0, \quad (\text{A.3})$$

$$K_\mu - \frac{1}{6}e^{-A}Z_{\mu\nu\lambda}F^{\nu\lambda} + e^A Y_{\mu\nu}\partial^\nu A = 0. \quad (\text{A.4})$$

Similarly the two-form identities are

$$F_{\mu\nu} - \frac{e^{-A}}{2}W_{\mu\nu\alpha\beta}F^{\alpha\beta} + 3Y_{\mu\nu} + 3(K_\mu\partial_\nu A - K_\nu\partial_\mu A) = 0, \quad (\text{A.5})$$

$$Z_{\mu\nu\alpha}\partial^\alpha A + \frac{1}{3}e^{-2A}(Y_{\mu\alpha}F_\nu{}^\alpha - Y_{\nu\alpha}F_\mu{}^\alpha) = 0. \quad (\text{A.6})$$

Let us present a 4-form equation also here which plays a crucial role when we check the gauge field equation of motion. One multiplies $\bar{\eta}\gamma_{\mu\nu\lambda\rho}$ to (2.12) and find

$$W - \frac{1}{3}Y \wedge F + \frac{1}{6}e^{-A+B} * F + e^A Z \wedge dA = 0. \quad (\text{A.7})$$

One can also first eliminate \not{F} in the equation and construct various spinor bilinear, i.e. start with

$$[\not{\partial}(A + 2B) + e^{-A} + 2ie^{-B}\gamma_7] \eta = 0. \quad (\text{A.8})$$

If we multiply $\bar{\eta}$ from left the real part gives

$$K^\mu \partial_\mu (A + 2B) = 0, \quad (\text{A.9})$$

and the imaginary part is

$$D = e^{-A+B}C/2. \quad (\text{A.10})$$

Below we list several of such algebraic relations.

$$L^\mu \partial_\mu (A + 2B) = 2e^{-B} C + e^{-A} D, \quad (\text{A.11})$$

$$L = \frac{e^B}{2} C d(A + 2B), \quad (\text{A.12})$$

$$Y' = -\frac{1}{2} e^{-A+B} Y - e^{-A} K \wedge L, \quad (\text{A.13})$$

$$Z' = +e^A d(A + 2B) \wedge Y', \quad (\text{A.14})$$

$$W = -e^{A-B} * Y + d(A + 2B) \wedge Z. \quad (\text{A.15})$$

B. Fierz identities

In this section we present a list of useful Fierz rearrangement identities for 6 dimensional commuting spinors. Our Killing spinor system is very similar to the 1/4-BPS solutions considered in [3], and we find the appendix very useful. Readers are referred to [3] for more identities and detailed derivations. In this section we will repeat some of the derivations in [3] and rephrase them in our convention for quick reference and self-sufficiency. We will also consider identities involving $\eta^T \eta$. In particular it will be shown how to derive (2.33).

In our convention γ_μ are all antisymmetric and generate 6 dimensional Clifford algebra. The chirality is defined in terms of

$$\gamma_7 = \gamma_{0123456}, \quad (\text{B.1})$$

and the positive(negative) chirality part of a spinor η is given as $\eta_\pm = \frac{1}{2}(1 \pm \gamma_7)\eta$.

The basic relation for Fierz rearrangement is (see (63) in [3])

$$\begin{aligned} \bar{\eta}_1 \eta_2 \bar{\eta}_3 \eta_4 &= \frac{1}{8} (\bar{\eta}_1 \eta_4 \bar{\eta}_3 \eta_2 + \bar{\eta}_1 \gamma_7 \eta_4 \bar{\eta}_3 \gamma_7 \eta_2) \\ &\quad - \frac{1}{16} (\bar{\eta}_1 \gamma_{\mu\nu} \eta_4 \bar{\eta}_3 \gamma^{\mu\nu} \eta_2 + \bar{\eta}_1 \gamma_{\mu\nu} \gamma_7 \eta_4 \bar{\eta}_3 \gamma^{\mu\nu} \gamma_7 \eta_2) \\ &\quad + \frac{1}{8} (\bar{\eta}_1 \gamma_\mu \eta_4 \bar{\eta}_3 \gamma^\mu \eta_2 - \bar{\eta}_1 \gamma_\mu \gamma_7 \eta_4 \bar{\eta}_3 \gamma^\mu \gamma_7 \eta_2) \\ &\quad - \frac{1}{96} (\bar{\eta}_1 \gamma_{\mu\nu\lambda} \eta_4 \bar{\eta}_3 \gamma^{\mu\nu\lambda} \eta_2 - \bar{\eta}_1 \gamma_{\mu\nu\lambda} \gamma_7 \eta_4 \bar{\eta}_3 \gamma^{\mu\nu\lambda} \gamma_7 \eta_2). \end{aligned} \quad (\text{B.2})$$

If we choose $\bar{\eta}_1 = \bar{\eta}_\pm \gamma_\mu$, $\eta_2 = \eta_\pm$, $\bar{\eta}_3 = \bar{\eta}_\pm$ and $\eta_4 = \gamma_\mu \eta_\pm$ one can derive

$$(K \pm L)^2 = 0, \quad (\text{B.3})$$

which in turn implies (2.30) and (2.31).

If one uses $\bar{\eta}_1 = \bar{\eta}_+$, $\eta_2 = \eta_-$, $\bar{\eta}_3 = \bar{\eta}_-$ and $\eta_4 = \eta_+$ we get

$$C^2 + D^2 = \frac{1}{4}(L^2 - K^2) + \frac{1}{48}(Z^2 - Z'^2). \quad (\text{B.4})$$

In order to prove (2.33), one chooses $\bar{\eta}_1 = \eta_+^T$, $\eta_2 = \eta_-$, $\bar{\eta}_3 = \bar{\eta}_-$ and $\eta_4 = \gamma_0 \eta_+^*$, to find

$$|\eta^T \eta|^2 = \frac{1}{4}(L^2 - K^2) - \frac{1}{48}(Z^2 - Z'^2), \quad (\text{B.5})$$

and as a result we can verify (2.33).

It is also desirable to compute the square of two-forms Y and ω . Choosing $\bar{\eta}_1 = \bar{\eta}_+, \eta_2 = \eta_-, \bar{\eta}_3 = \bar{\eta}_+$ and $\eta_4 = \eta_-$, we get

$$C^2 - D^2 = \frac{1}{6}(Y^2 - Y'^2), \tag{B.6}$$

$$CD = \frac{1}{6}Y \cdot Y', \tag{B.7}$$

and we also consider $\bar{\eta}_1 = \eta_+^T, \eta_2 = \eta_-, \bar{\eta}_3 = \bar{\eta}_+$ and $\eta_4 = \gamma_0 \eta_-^*$ and find

$$|\eta^T \eta|^2 = -\frac{1}{4}(C^2 + D^2) + \frac{1}{8}(Y^2 + Y'^2). \tag{B.8}$$

We can thus conclude,

$$Y^2 = 4C^2 - 2D^2 + 4|\eta^T \eta|^2, \quad Y'^2 = -2C^2 + 4D^2 + 4|\eta^T \eta|^2, \tag{B.9}$$

We can also use $\bar{\eta}_1 = \eta_+^T, \eta_2 = \gamma_0 \eta_-^*, \bar{\eta}_3 = \bar{\eta}_+$ and $\eta_4 = \eta_-$ to obtain

$$|\eta^T \eta|^2 = \frac{1}{2}\omega \cdot \omega^* - 4(C^2 + D^2). \tag{B.10}$$

In order to see the decomposition of 6 dimensional tensors in terms of 4 dimensional ones, we need to compute their contractions with L and K . The results are given in (75), (76), (77) and (78) of Donos. In our notation they become

$$i_K Y = DL, \tag{B.11}$$

$$i_L Y = DK, \tag{B.12}$$

$$i_K Y' = CL, \tag{B.13}$$

$$i_L Y' = CK. \tag{B.14}$$

To compute the contraction of ω with one-forms we consider $\bar{\eta}_1 = \bar{\eta}_\pm \gamma_\mu, \eta_2 = \eta_\pm, \bar{\eta}_3 = \eta_\pm^T$ and $\eta_4 = \gamma^\mu \gamma_\nu \eta_\mp$ and find

$$\bar{\eta} \gamma_\mu (1 \pm \gamma_\tau) \eta \cdot \eta^T \gamma^\mu \gamma_\nu (1 \mp \gamma_\tau) \eta = 0, \tag{B.15}$$

which leads to

$$i_K \omega = \frac{1}{2}(\eta^T \eta)L, \tag{B.16}$$

$$i_L \omega = \frac{1}{2}(\eta^T \eta)K. \tag{B.17}$$

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